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Nonlinear Control and Model Analysis of Trirotor UAV Model

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Abstract: This paper reveals the minimal number of inputs for hovering control from the perspective of nonlinear control system theory. Then, a nonlinear controller is designed to obtain meaningful motion for an underactuated UAV with the consideration of the limitation acknowledged through the model analysis. Throughout this paper, a particular UAV model having three rotors is considered as a controlled object. First, nonlinear state equations are computed with valid assumptions. Next, the model analysis reveals that hovering control is impossible for not only the proposed model but also a generic three-inputs model through the argument of Locally Asymptotically Stabilizability(LAS). Finally, numerical simulation confirms that position control is realizable via Output Zeroing Control along with preferable attitude.

Keywords: UAV, Robotics, Underactuated System, Nonlinear Analysis, Locally Asymptotically Stabilizability, Nonlinear Control, Output Zeroing Control, System Failure

1. INTRODUCTION

Various researches regarding Unmmaned Aerial Vehicle (UAV) have gathered researcher's attention with the aim of practical usage; Mars exploration, inspection of building and bridges, wildlife monitoring and other innovative applications. The focus of UAV researches covers a broad range of topics from the perspective of control engineering. For example, trajectory tracking control problem by quadrotor helicopter has been done for more complex mission. (Gabriel M. Hoffmann and Tomlin (2007), Voos and Bou-Ammar (2010)) For the sake of reduction of weight, volume and energy consumption, Trirotor model has been suggested. (Sergio Salazar-Cruz and Fantoni (2006), Penghui Fan and Cai (2010)) In addition, control for failure case has highly contributed to safety aspect. (I.Byrnes and Isidori (1991))

Generic VTOL UAV nonlinear model has 6 DOF, translational and rotational representation in three dimensional space, whereas it usually has less number of inputs, which is called an underactuated system. That is, UAV is considered as an interesting system from the perspective of study of nonlinear control system. Specifically, questions have arisen as for the minimal number of inputs for hovering control and the achievable control tasks under insufficient inputs. The first question is equivalent to the argument of the Locally Asymptotically Stabilizability(LAS) of a nonlinear system. The second question is challenged by controllability analysis and nonlinear controller design. Remark that this theoretical approach study has potential to contribute to failure case of UAV for the sake of collision avoidance in practical usage.

In this paper, a new type of Trirotor system is mainly utilized for the analysis of three-input UAV model. The proposed Trirotor model owns literally only three inputs, which means this does not own any other input such as servo motors as other Trirotor models have. In section 2, two nonlinear state equations are derived based on Euler angle and quaternion, respectively. Then, the first main contribution of this paper is shown in section 3, that is, three-input model does not own LAS. However, the analysis of controllabilily indicates that other meaningful motions can be realized. Taking the results of model analysis into account, a nonlinear controller is designed in section 4 with the brief introduction of MIMO Output Zeroing Control. Finally, numerical simulation verifies the performance of position control along with the upward posture.

2. MODELING

In this section, the two nonlinear state equations, Euler angle based model and quaternion based model, are derived for the proposed Trirotor UAV model shown in Fig.1 with the consideration of results by T. Cheviron and Plestan (2009). One of the main features of this model is the three rotors which is installed on three skew axes. One note here is that this model does not own other inputs such as a servo motor to control tilting angle as modeled. Since the main intention of this paper is the analysis and control design

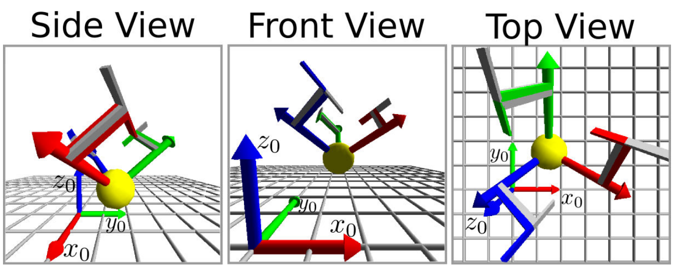


Fig. 1. Proposed UAV Model from Different Angle

Table 1. Physical Variables

Σ_o	:	inertial coordinate system
Σ_r	:	relative coordinate system
$\boldsymbol{\xi}$	:	position of the center of gravity of robot (x, y, z)
$\boldsymbol{\eta}$	:	posture angle : Euler angle (ϕ, θ, ψ)
$\boldsymbol{\omega}_r$	:	angular velocity vector defined on Σ_r
$\dot{\theta}_{pi}$	:	angular velocity of ith rotor(propeller)
$\dot{\boldsymbol{\theta}}_p$	:	diagonal matrix of $\dot{\theta}_{pi}$, diag$\{\dot{\theta}_{p1}, \dot{\theta}_{p2}, \dot{\theta}_{p3}\}$
F_{ri}	:	force to the $i(x_r, y_r, z_r)$ direction on Σ_r
$\boldsymbol{F}_r$	:	force vector on Σ_r, diag$\{F_{rx}, F_{ry}, F_{rz}\}$
N_{F_i}	:	moment of force generated by F_i
N_{τ_i}	:	counteraction generated by the rotation of rotor i
$\boldsymbol{N}_r$	:	total moment vector on Σ_r
$\boldsymbol{u}$	:	torque input to each motors $[u_3, u_1, u_2]^T$

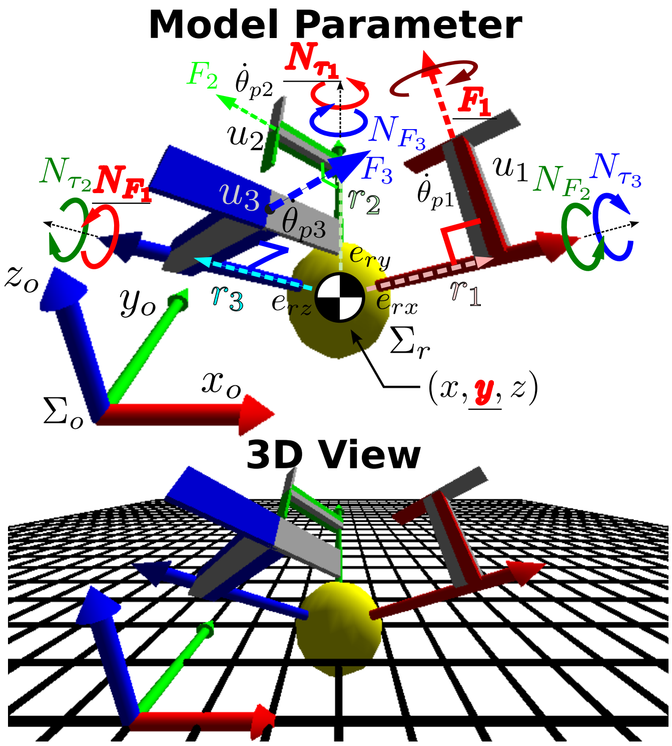


Fig. 2. Overview of UAV Model having Three Rotors

for an underactuated UAV model, this model can be a good starting example.

Here, the behavior of this model is described roughly along with Fig.2. As is shown in bold red letters in Fig.2, the rotation of rotor 1 generates the translational force in positive y direction. Plus, this rotation generates two different moments, positive rotation about the axis of y and negative rotation about the axis of z. Consequently, two moments on each axis cancel each other to suppress the attitude variation. The more details about model assumption are given in the next section.

2.1 Model Assumption of Trirotor

Input to UAV The system inputs to this system are assumed to be torque inputs driven by DC motors which are connected to three rotors. This assumption means the dynamics of rotor is explicitly taken into account with the consideration of the back electromotive force and viscous force by

$$\boldsymbol{J}_p \ddot{\boldsymbol{\theta}}_p + \boldsymbol{D}_p \dot{\boldsymbol{\theta}}_p = \boldsymbol{u} \tag{1}$$

Table 2. Physical Parameters

m	:	mass of the proposed Trirotor model
$\boldsymbol{J}_r$	:	inertia tensor defined on Σ_r(constant)
$\boldsymbol{D}_r$	:	viscous matrix by air friction defined on Σ_r
$\boldsymbol{J}_p$	:	diag$\{J_{p1}, J_{p2}, J_{p3}\}$, J_{pi} : rotary inertia of ith rotor
$\boldsymbol{D}_p$	:	diag$\{D_{p1}, D_{p2}, D_{p3}\}$, D_{pi} : coefficient of viscosity
k_{pi}	:	coefficient of property of ith rotor
$\boldsymbol{r}_i$	:	vector from center of mass to operating point of F_i
C_T	:	thrust coefficient of rotor
ρ	:	air density
πR^2	:	disk area by rotor rotation
$\dot{\theta}_p R$	:	circumferential speed of the rotor tip
g	:	gravitational acceleration

.

Counteraction against Input Torque The counteraction against input torque can not be neglected for this system, since the inertia of this vehicle is assumed to be relatively small to the input torque. Therefore, all of the moments vector generated by the three inputs are described as

$$\boldsymbol{N}_r = \begin{bmatrix} N_{F_2} - N_{\tau_3} \\ N_{F_3} - N_{\tau_1} \\ N_{F_1} - N_{\tau_2} \end{bmatrix} = \begin{bmatrix} (r_2 \times F_2) - u_3 \\ (r_3 \times F_3) - u_1 \\ (r_1 \times F_1) - u_2 \end{bmatrix} \tag{2}$$

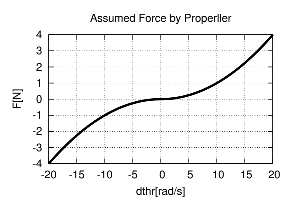
. Note this expression is defined on the relative coordinate system Σ_r. This equation shows that the two moments cancel each other, if the positive translational force is generated by the positive rotation.

Propulsion Force generated by Rotors Each translational force generated by the three rotors is proportional to square of the angular velocity of the rotors. This assumption is based on aerodynamics. These constant parameters in

$$F = C_T \rho \pi R^2 (\dot{\theta}_p R)^2 \tag{3}$$

represent the property of the rotor. The physical meanings of the constants are shown in Table 2. Remark that this parameter is adjustable by mechanical design. In this paper, the constant parameter with respect to rotor i is represented by k_{pi}. Because this rotor has only one rotation inputs, the positive rotation generates positive translational force and vice versa. Based on these assumptions, the thrust vector $\boldsymbol{F}_r$ defined on the relative coordination system Σ_r is shown by

$$\begin{aligned} \boldsymbol{F}_r &= [F_{r3}, F_{r1}, F_{r2}]^T \\ &= \begin{bmatrix} k_{p3}\mathrm{sgn}(\dot{\theta}_{p3})\dot{\theta}_{p3}^2 \\ k_{p1}\mathrm{sgn}(\dot{\theta}_{p1})\dot{\theta}_{p1}^2 \\ k_{p2}\mathrm{sgn}(\dot{\theta}_{p2})\dot{\theta}_{p2}^2 \end{bmatrix} \end{aligned} \tag{4}$$



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Here, the relative degree regarding both position from inputs and attitude from them are analysed, respectively. First, it is easily shown that the relative degree of position from system input is 3 by taking derivative of the force, $\frac{\mathrm{d}}{\mathrm{d}t}F_{ri} = 2k_{p_i}\mathrm{sgn}(\dot{\theta}_{pi})\dot{\theta}_{pi}\ddot{\theta}_{pi}$. Second, the relative degree of attitude from system input is 2, since the counteraction of input torque directly affects attitude. The effect of this difference contributes to the system representation.

Air Friction caused by Body Rotation It is also assumed that the air friction caused by body rotation generates viscous term to attitude dynamics. This viscous term physically means the rotation velocity converges, if there remains constant moment to the system.

2.2 Nonlinear State Equations

In this section, the nonlinear equations of the proposed Trirotor UAV model are derived with the previous assumptions. Since there exist various types of attitude representations which have different features, it is important to use different system realization depending on the purpose. In this paper, Euler angle and quaternion are considered for the attitude representation. Euler angle is relatively beneficial with the respect to system analysis, whereas quaternion has advantage for controller design. The former explicitly exhibits physical property, whereas the latter has no singular points that the former has.($\cos\theta = 0$, where θ is pitch angle). Therefore, Euler angle representation is used for system analysis, and quaternion model is for controller design.

Euler Angle Model Euler angle model is derived via Euler-Lagrange method. First, let $\boldsymbol{q}_{le}$ denote the state variables,

$$\boldsymbol{q}_{le} = [\boldsymbol{\xi}^T, \dot{\boldsymbol{\xi}}^T, \boldsymbol{\eta}^T, \dot{\boldsymbol{\eta}}^T, \dot{\boldsymbol{\theta}}_p^T]^T \in \mathbb{R}^{12} \times \mathbb{S}^3 \tag{5}$$

where each variable is defined in Table 1. Since the dynamics of the rotor is explicitly considered in this model, not only position and attitude dynamics but also velocity of the rotor is included. In total, 15 state variables are used for this realization. Second, let $\boldsymbol{R}(\theta, \phi, \psi)$ be the rotation matrix defined by the roll-pitch-yaw representation as is shown in

$$\begin{aligned} \boldsymbol{R}(\theta,\phi,\psi) &= \boldsymbol{R}_z(\psi)\boldsymbol{R}_y(\theta)\boldsymbol{R}_x(\phi) \\ &= \begin{bmatrix} C_\psi C_\theta & -C_\phi S_\psi + C_\psi S_\phi S_\theta & S_\phi S_\psi + C_\phi C_\psi S_\theta \\ C_\theta S_\psi & C_\phi C_\psi + S_\phi S_\psi S_\theta & -C_\psi S_\phi + C_\phi S_\psi S_\theta \\ -S_\theta & C_\theta S_\phi & C_\phi C_\theta \end{bmatrix} \end{aligned} \tag{6}$$

. Here, $S_* = \sin *$ and $C_* = \cos *$ are used for simplicity. This rotation matrix is the transformation of the vector defined on relative coordination system Σ_r, $\boldsymbol{\omega}_r = [\omega_{1r}, \omega_{2r}, \omega_{3r}]^T$, to global coordination system Σ_0, $\boldsymbol{\omega}_0 = [\omega_{10}, \omega_{20}, \omega_{30}]$. Thus,

$$\boldsymbol{\omega}_0 = \boldsymbol{R}(\theta,\phi,\psi)\boldsymbol{\omega}_r \tag{7}$$

is satisfied. It is also gained that how the vector $\boldsymbol{\omega}_r$ is transformed to the velocity vector of Euler angle. Considering each rotation matrices, $(\boldsymbol{R}_z(\psi), \boldsymbol{R}_y(\theta), \boldsymbol{R}_x(\phi))$, the relation between euler angle and angular velocity is argued by

$$\begin{aligned} \boldsymbol{\omega}_r &= \boldsymbol{R}_x(\phi)^T \boldsymbol{R}_y(\theta)^T \begin{bmatrix} 0 \\ 0 \\ \dot\psi \end{bmatrix} + \boldsymbol{R}_x(\phi)^T \begin{bmatrix} 0 \\ \dot\theta \\ 0 \end{bmatrix} + \begin{bmatrix} \dot\phi \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} = \boldsymbol{T}(\phi,\theta) \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} \end{aligned} \tag{8}$$

. Remark that the equivalent effect of the moment vector in (2) to Euler angle space $\boldsymbol{N}_\eta$ is computed via principle of virtual work as is shown in

$$\begin{aligned} (\boldsymbol{\delta}_{\boldsymbol{\omega}_r})^T \boldsymbol{N}_r &= (\boldsymbol{\delta}_\eta)^T \boldsymbol{N}_\eta \\ \Leftrightarrow \quad \boldsymbol{\omega}_r^T \boldsymbol{N}_r &= \dot{\boldsymbol{\eta}}^T \boldsymbol{N}_\eta \\ \Leftrightarrow \boldsymbol{T}(\phi,\theta)^T \boldsymbol{N}_r &= \boldsymbol{N}_\eta \end{aligned} \tag{9}$$

. Third, the Lagrangian of this system is

$$L = L(z, \phi, \theta, \dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\eta}}, \dot{\boldsymbol{\theta}}_p) = K - U \tag{10}$$

, since the total kinetic, potential and dissipative energy are computed as

$$K = \frac{1}{2} m \dot{\boldsymbol{\xi}}^T \dot{\boldsymbol{\xi}} + \frac{1}{2}\boldsymbol{\omega}_r^T(\phi,\theta,\dot{\boldsymbol{\eta}}) \boldsymbol{J}_r \boldsymbol{\omega}_r(\phi,\theta,\dot{\boldsymbol{\eta}}) + \frac{1}{2}\dot{\boldsymbol{\theta}}_p^T \boldsymbol{J}_p \dot{\boldsymbol{\theta}}_p \tag{11}$$

$$U = mgz \tag{12}$$

$$D = \frac{1}{2}\boldsymbol{\omega}_r^T(\phi,\theta,\dot{\boldsymbol{\eta}}) \boldsymbol{D}_r \boldsymbol{\omega}_r(\phi,\theta,\dot{\boldsymbol{\eta}}) + \frac{1}{2}\dot{\boldsymbol{\theta}}_p^T \boldsymbol{D}_p \dot{\boldsymbol{\theta}}_p \tag{13}$$

. (10) shows that the Lagrangian of this system does not depend on the yaw angle, ψ. This feature will be important factor in the next section, system analysis.

Finally, the dynamics of this system is given in

$$\begin{aligned} \boldsymbol{M}\ddot{\boldsymbol{\xi}} &= -\boldsymbol{C}_\xi + \boldsymbol{R}(\boldsymbol{\eta})\boldsymbol{F}_r \\ \boldsymbol{J}\ddot{\boldsymbol{\eta}} &= -\boldsymbol{C}_\eta(\phi,\theta,\dot{\boldsymbol{\eta}}) + \boldsymbol{T}^T(\phi,\theta)\boldsymbol{N}_r. \end{aligned} \tag{14}$$

Consequently, the 15 dimensional nonlinear state equation

$$\dot{\boldsymbol{q}}_{le} = \boldsymbol{f}_{le}(\boldsymbol{q}_{le}) + \boldsymbol{g}_{le}(\boldsymbol{q}_{le})\boldsymbol{u} \tag{15}$$

$$\text{where } \boldsymbol{f}_{le}(\boldsymbol{q}_{le}) \in \mathbb{R}^{15\times 1}, \boldsymbol{g}_{le}(\boldsymbol{q}_{le}) \in \mathbb{R}^{15\times 3}$$

is obtained. The details of $\boldsymbol{f}_{le}(\boldsymbol{q}_{le})$ and $\boldsymbol{g}_{le}(\boldsymbol{q}_{le})$ are omitted due to the limitation of the space.

Quaternion Model The translational dynamics is same as the result argued in previous section. The focus of this section is the attitude dynamics using quaternion. The total dynamics is derived by the combination of the quaternion dynamics and Euler's equations of motion. The definition of quaternion is shown as follows,

$$\boldsymbol{q}_{eq} = [\boldsymbol{\xi}^T, \dot{\boldsymbol{\xi}}^T, \boldsymbol{e}^T, \boldsymbol{\omega}_r^T, \dot{\boldsymbol{\theta}}_p^T]^T \in \mathbb{R}^{12} \times \mathbb{S}^3 \tag{16}$$

$$\begin{aligned} \boldsymbol{e} &= [e_0, e_1, e_2, e_3]^T \in \mathbb{S}^3 \\ &= [\cos\frac{\theta}{2}, l\sin\frac{\theta}{2}, m\sin\frac{\theta}{2}, n\sin\frac{\theta}{2}]^T \end{aligned} \tag{17}$$

$$\boldsymbol{\kappa} = [l, m, n]^T \tag{18}$$

where, $\boldsymbol{e}$ is called quaternion or Euler parameter, $\boldsymbol{\kappa}$ is normalized axis of rotation, θ is the angle about $\boldsymbol{\kappa}$, and $\boldsymbol{q}_{eq}$ is the state variable vector. Although the number of the state variables are 16, this system representation has 15 dimension, because quaternion includes redundancy in its expression. The total attitude dynamics is represented by

$$\dot{\boldsymbol{e}} = \frac{1}{2}\left[\boldsymbol{e} \begin{pmatrix} -e_1 & -e_2 & -e_3 \\ e_0 & -e_3 & e_2 \\ e_3 & e_0 & -e_1 \\ -e_2 & e_1 & e_0 \end{pmatrix}\right] \begin{bmatrix} -2(\boldsymbol{e}^T\boldsymbol{e} - 1) \\ \boldsymbol{\omega}_r \end{bmatrix} \tag{19}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{J}_r(\boldsymbol{\omega}_r) = -\boldsymbol{\omega}_r \times (\boldsymbol{J}_r\boldsymbol{\omega}_r) + \boldsymbol{N}_r \tag{20}$$

. In total, the quaternion based nonlinear equation model is explicitly described by

$$\dot{\boldsymbol{q}}_{eq} = \boldsymbol{f}_{eq}(\boldsymbol{q}_{eq}) + \boldsymbol{g}_{eq}(\boldsymbol{q}_{eq})\boldsymbol{u} \tag{21}$$

$$\text{where } \boldsymbol{f}_{eq}(\boldsymbol{q}_{eq}) \in \boldsymbol{R}^{16\times 1}, \boldsymbol{g}_{eq}(\boldsymbol{q}_{eq}) \in \boldsymbol{R}^{16\times 3}$$

. Contrary to (15), there does not exist singular point in this model realization.

3. MODEL ANALYSIS

In this section, the property of the model is identified in terms of nonlinear system theory, specifically, LAS(Locally

Asymptotically Stabilizability) and controllability. As is mentioned in previous section, Euler angle model given in (15) is used for model analysis.

3.1 Locally Asymptotically Stabilizability(LAS)

The idea of locally asymptotically stabilizability(LAS) for UAV system is equivalent to the argument of whether hovering control is mathematically possible or not. The main procedure to reach the conclusion starts with the analysis of equilibrium set, and then the result gained in Ishikawa and Sampei (1998) is applied. Consequently, it is shown to be impossible to obtain hovering control, stabilization of position and attitude simultaneously, via state feedback control in the case of UAV having only three-rotor inputs.

First of all, let us cite the significantly important theorem. M.Ishikawa and M.Sampei revealed a necessary condition for LAS that is *"A system is not feedback stabilizable if the number of inputs is less than the dimension of the equilibria submanifold."*. That is, it is important to analyze the dimension of the equilibrium submanifold of UAV system in order to check LAS.

Next, the main result is shown for the case of the proposed UAV model in previous section.

Theorem 1. It is impossible to locally asymptotically stabilize the UAV model of (15) by state feedback control.

Proof. Obviously, the existence of one or more equilibrium points is necessary for locally asymptotically stabilizability. Thus, it is investigated at first whether equilibrium exists or not. If there exists at least one equilibrium point,

$$\dot{\boldsymbol{q}}_{le} = \boldsymbol{f}_{le}(\boldsymbol{q}_{le}) + \boldsymbol{g}_{le}(\boldsymbol{q}_{le})\boldsymbol{u}|_{\dot{\boldsymbol{\xi}}=\boldsymbol{0},\dot{\boldsymbol{\eta}}=\boldsymbol{0},\boldsymbol{u}=\boldsymbol{\tau}(\dot{\boldsymbol{\theta}}_p)} = \boldsymbol{0} \quad (22)$$

needs to be satisfied, where (22) is obtained by the assumption at equlibrium point, $\dot{\boldsymbol{\xi}} = \boldsymbol{0}, \dot{\boldsymbol{\eta}} = \boldsymbol{0}$ and $\boldsymbol{u} = \boldsymbol{\tau}(\dot{\boldsymbol{\theta}}_p)$. Remark $\boldsymbol{u} \neq \boldsymbol{0}$ at equilibrium point for this model. These constraints reduce the 15 nonlinear equations to the essential 6 equations shown in

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} \gamma_1(\phi,\theta,\psi,\dot{\theta}_{p1},\dot{\theta}_{p2},\dot{\theta}_{p3}) \\ \gamma_2(\phi,\theta,\psi,\dot{\theta}_{p1},\dot{\theta}_{p2},\dot{\theta}_{p3}) \\ \gamma_3(\phi,\theta,\dot{\theta}_{p1},\dot{\theta}_{p2},\dot{\theta}_{p3}) - g \\ \gamma_4(\phi,\theta,\dot{\theta}_{p1},\dot{\theta}_{p2},\dot{\theta}_{p3}) \\ \gamma_5(\phi,\theta,\dot{\theta}_{p1},\dot{\theta}_{p2},\dot{\theta}_{p3}) \\ \gamma_6(\phi,\theta,\dot{\theta}_{p1},\dot{\theta}_{p2},\dot{\theta}_{p3}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (23)$$

. Apparently, position $\boldsymbol{\xi}$ does not affect the equations, which means three dimensional position is counted in equilibrium submanifold. In addition, as for the yaw angle ψ, the precise equations regarding $\ddot{x} = 0$ and $\ddot{y} = 0$ turns out to be

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} S_\psi & C_\psi \\ -C_\psi & S_\psi \end{bmatrix} \begin{bmatrix} \dfrac{k\,S_\phi\,\dot{\theta}_{p2}^2}{m} - \dfrac{k\,C_\phi\,\dot{\theta}_{p1}^2}{m} \\ \dfrac{k\,C_\phi\,\dot{\theta}_{p2}^2\,S_\theta}{m} + \dfrac{k\,S_\phi\,\dot{\theta}_{p1}^2\,S_\theta}{m} + \dfrac{k\,\dot{\theta}_{p3}^2\,C_\theta}{m} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (24)$$

. This means that the arbitrary value of yaw angle ψ is always eliminated, which means ψ is included in equilibrium submanifold as well as position. Unless there exist 5 solutions, $(\phi^*, \theta^*, \theta_{p1}^*, \theta_{p2}^*, \theta_{p3}^*)$, which satisfy 6 equations of (23), equilibrium point does never exist.

(i) No solution case to simultaneous equation (23)
As is stated before, no equilibrium point means no LAS. In this case, obviously hovering control is impossible.

(ii) Existence of solution case
There is a chance that there exists an equilibrium point under model parameter constraint such as the case that the property of the rotor k_{pi} satisfies

$$k_{pi} = \frac{\sqrt{3}d_r^2}{mgl^2} \quad (25)$$

. Since this parameter depends on the mechanical design of the rotor, it is adjustable to realize (25) as is explained in section 2.1. However, the equilibrium set $N^\flat$ turns out to be $N^\flat = \{x, y, z, \psi\} \in \mathbb{R}^3 \times \mathbb{S}^1$, whereas $\boldsymbol{u} \in \mathbb{R}^3$. Because the number of inputs is less than the dimension of $N^\flat$, it is concluded that there does not exist continuous state feedback that leads to locally asymptotic stabilization.

□

In addition to *Theorem 1*, the general case is derived based on the extension of the previous analysis.

Theorem 2. Assume that UAV is composed of only a rigid body and propellers, then it is impossible to locally asymptotically stabilize a generic UAV model having three rotors inputs via continuous state feedback.

Proof. The main procedure of the following proof is same as that of *Theorem 1*. With the assumption, the total kinetic energy is always computed by the translational, rotational energy of rigid body and energy of propellers. Plus, the total potential energy is determined by gravitational potential energy. Thus, the energy computation of (10) and (13) is applied for the generic case as well. According to the Euler-Lagrange differential equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{\xi}}}\right) - \frac{\partial L}{\partial \boldsymbol{\xi}} + \frac{\partial D}{\partial \dot{\boldsymbol{\xi}}} = \boldsymbol{R}(\theta,\phi,\psi)\boldsymbol{F}_r(\dot{\boldsymbol{\theta}}_p) \quad (26)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{\eta}}}\right) - \frac{\partial L}{\partial \boldsymbol{\eta}} + \frac{\partial D}{\partial \dot{\boldsymbol{\eta}}} = \boldsymbol{T}^T(\theta,\phi)\boldsymbol{N}_r(\boldsymbol{u},\dot{\boldsymbol{\theta}}_p) \quad (27)$$

, it is eventually concluded that the effect of ψ only appears in rotation matrix (6) in generic case. Hence, it is proved that yaw angle ψ is always included in equilibrium submanifold $N^\flat$, since the effect of ψ is separated as is in (24) against any $\boldsymbol{F}_r$. Thus, the number of inputs, 3, is always less than dimension of $N^\flat$.

□

In other words, *Theorem 2* says the variation of any model having three inputs does not affect LAS; hovering control via continuous state feedback is unrealizable, no matter where the rotor is installed or no matter what inertia tensor $\boldsymbol{J}_r$ the robot has, unless it has more than 4 inputs. This result might contribute to the knowledge of the limitation with respect to the mechanical design of UAV which aims for hovering control.

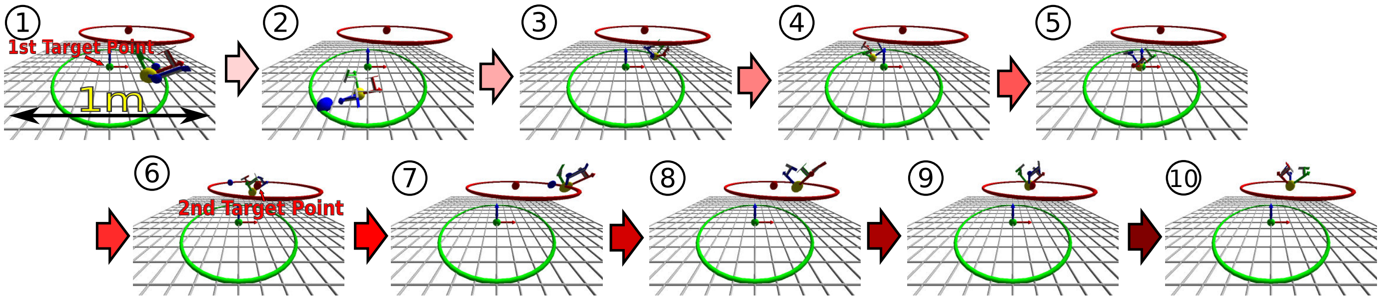

Fig. 3. Position and Attitude Control via Output Zeroing Control; Spiral Motion Control

3.2 Controllability

Reflecting the previous result, the dynamics without ψ is considered for the analysis of controllability.

In the nonlinear system theory, various types of notion in terms of controllability has been proposed. Especially for the case of asymmetric affine control systems, there is the analytic method to argue the controllability that has been suggested by Sussman such as STLC(Small Time Locally Controllability). In this paper, our intention is to investigate the controllability against the subspace that neglects the dynamics of ψ. However, there is no analytical method to check the controllablity against subspace of entire system, as far as the authors know. Therefore, in this section, the local controllablity is discussed based on an approximately linearized system around one candidate of equilibrium point

$$\dot{\tilde{\boldsymbol{q}}}_{le}^{*} = \left[\boldsymbol{O}^{1\times 6}, \frac{\pi}{4}, \tan^{-1}\left(\frac{-1.0}{\sqrt{2.0}}\right), \boldsymbol{O}^{1\times 3}, f^{*}, f^{*}, f^{*}\right]^{T} \quad (28)$$

, which corresponds to upward posture shown in Fig.2. The controllability is discussed as

$$\boldsymbol{A} = \left.\frac{\partial \dot{\tilde{\boldsymbol{q}}}_{le}(\tilde{\boldsymbol{q}}_{le}, \boldsymbol{u})}{\partial \boldsymbol{q}_{le}}\right|_{\substack{\boldsymbol{q}_{le} = \dot{\tilde{\boldsymbol{q}}}_{le}^{*} \\ \boldsymbol{u} = \boldsymbol{u}^{*}}}, \boldsymbol{B} = \left.\frac{\partial \dot{\tilde{\boldsymbol{q}}}_{le}(\tilde{\boldsymbol{q}}_{le}, \boldsymbol{u})}{\partial \boldsymbol{u}}\right|_{\substack{\boldsymbol{q}_{le} = \dot{\tilde{\boldsymbol{q}}}_{le}^{*} \\ \boldsymbol{u} = \boldsymbol{u}^{*}}}$$

$$f^{*} = \sqrt{\frac{gm}{\sqrt{3}k_{pi}}} \quad \Rightarrow \quad rank\left(\mathcal{C}(\boldsymbol{A}, \boldsymbol{B})\right) = 14 \quad (29)$$

. This result shows that, at least, all the state variables except for ψ is locally controllable around $\dot{\tilde{\boldsymbol{q}}}_{le}^{*}$ and $\boldsymbol{u}^{*}$, because the controllability matrix in (29) has full rank.

4. CONTROLLER DESIGN

Although the hovering control is proved to be impossible, there remains the possibility to achieve meaningful motion with neglecting certain dynamics. In this section, a nonlinear controller to the underactuated UAV system is proposed, aiming for position control along with upward posture. First, the design of Output Zeroing control is briefly explained. Second, numerical simulation shows the performance of the position control. In addition, it demonstrates that this controller is also intended to control roll angle and pitch angle indirectly. For the practical purposes, this design contributes to the control problem of failure situation of other ordinal UAVs for safe landing.

4.1 Output Zeroing Control MIMO case

This section explains the design procedure of the Output Zeroing controller design for three inputs case. Output Zeroing control is based on input-output linearization. Unless there exists strictly linearizable transforming coordination, this method converts nonlinear system (21) into the combination of linear subsystem (38) and nonlinear subsystem (42). First, define new coordination system ($\boldsymbol{z} = \boldsymbol{\Phi}(\boldsymbol{q}_{le}) = [\phi_1, ..., \phi_n]^T$). Second, let output functions and the derivative of them be the new states,

$$\phi_1^i = h_i,\ \phi_2^i = L_f h_i,\ \cdots,\ \phi_{r_i}^i = L_f^{r_i-1} h_i \quad (30)$$

$$\boldsymbol{\sigma} = [\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2, \boldsymbol{\sigma}^3]^T,\ \boldsymbol{\sigma}^i = [\phi_1^i, \cdots, \phi_{r_i}^i]^T \quad (31)$$

$$\boldsymbol{\zeta} = [\phi_{r+1}^i, \phi_{r+2}^i, \cdots, \phi_n^i]^T,\ r = r_1 + r_2 + r_3 \quad (32)$$

where h_i is ith output function and r_i is the relative degree regarding ith output function. Here, $\boldsymbol{\zeta}$ needs to be chosen so that the transformation $\boldsymbol{\Phi}(\boldsymbol{q}_{le})$ satisfies a *diffeomorphism*. Moreover, the nonlinear system needs to hold *vector relative degree*. Third, superimpose feedback input defined by

$$\boldsymbol{u} = \boldsymbol{\alpha}^{-1}(\boldsymbol{\sigma}, \boldsymbol{\zeta})\left(\boldsymbol{\beta}(\boldsymbol{\sigma}, \boldsymbol{\zeta}) - \boldsymbol{v}\right) \quad (33)$$

where

$$\boldsymbol{\alpha}(\boldsymbol{\sigma}, \boldsymbol{\zeta}) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},\ \boldsymbol{\beta}(\boldsymbol{\sigma}, \boldsymbol{\zeta}) = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad (34)$$

$$a_{ij}(\boldsymbol{\sigma}, \boldsymbol{\zeta}) = L_{g_{eq_j}} L_{f_{eq}}^{r_i-1} h_i(\boldsymbol{\Phi}^{-1}(\boldsymbol{\sigma}, \boldsymbol{\zeta})) \quad (35)$$

$$b_i(\boldsymbol{\sigma}, \boldsymbol{\zeta}) = L_{f_{eq}}^{r_i} h_i(\boldsymbol{\Phi}^{-1}(\boldsymbol{\sigma}, \boldsymbol{\zeta})) \quad (36)$$

$$\boldsymbol{h} = \left[h_1(\boldsymbol{q}_{eq}),\ h_2(\boldsymbol{q}_{eq}),\ h_3(\boldsymbol{q}_{eq})\right]^T \quad (37)$$

This results in transforming original system into r dimensional linear subsystem

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{A}\boldsymbol{\sigma} + \boldsymbol{B}\boldsymbol{v} \quad (38)$$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{O}^{r_1\times r_2} & \boldsymbol{O}^{r_1\times r_3} \\ \boldsymbol{O}^{r_2\times r_1} & \boldsymbol{A}_2 & \boldsymbol{O}^{r_2\times r_3} \\ \boldsymbol{O}^{r_3\times r_1} & \boldsymbol{O}^{r_3\times r_2} & \boldsymbol{A}_3 \end{bmatrix} \quad (39)$$

$$\boldsymbol{A}_i = \begin{bmatrix} \boldsymbol{O}^{(r_i-1)\times 1} & \boldsymbol{I}^{(r_i-1)\times(r_i-1)} \\ \boldsymbol{O}^{1\times 1} & \boldsymbol{I}^{1\times(r_i-1)} \end{bmatrix} \quad (40)$$

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{O}^{r_1\times 1} & \boldsymbol{O}^{r_1\times 1} \\ \boldsymbol{O}^{r_2\times 1} & \boldsymbol{B}_2 & \boldsymbol{O}^{r_2\times 1} \\ \boldsymbol{O}^{r_3\times 1} & \boldsymbol{O}^{r_3\times 1} & \boldsymbol{B}_3 \end{bmatrix},\ \boldsymbol{B}_i = \begin{bmatrix} \boldsymbol{O}^{(r_i-1)\times 1} \\ \boldsymbol{I}^{1\times 1} \end{bmatrix} \quad (41)$$

and $n - r$ dimensional unobservable nonlinear subsystem

$$\dot{\boldsymbol{\zeta}} = \boldsymbol{\Gamma}(\boldsymbol{\sigma}, \boldsymbol{\zeta}) \quad (42)$$

. Here, $\boldsymbol{v}$ can be designed by linear control theory such as LQR control. Consequently, the output functions and derivative of them converge to zero, whereas the nonlinear subspace is represented by $\dot{\boldsymbol{\zeta}} = \boldsymbol{\Gamma}(\boldsymbol{0}, \boldsymbol{\zeta})$, when output zeroing is realized. This remaining dynamics is called *Zero Dynamics*. The notable features of Output Zeroing Control are as follows.

Table 3. Model Parameters

m :	mass of robot	: 0.50 [kg]
$\lvert r_i \rvert$:	distance between CofG and rotor	: 15 [cm]
g :	gravitational acceleration	: 9.8 [m/s^2]
A :	initial radius of spiral motion	: 50 [cm]
ρ :	decrease ratio of radius	: 0.5
δ :	inflection point of radius variation	: 10.0[s]

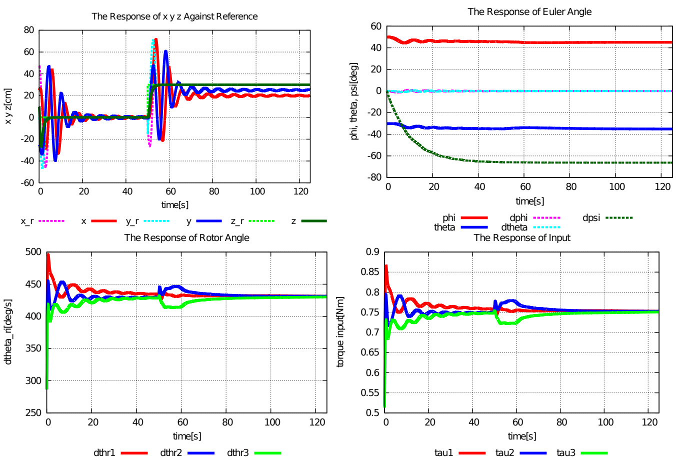

Fig. 4. Simulation Results of Maneuver Control

(1) Intuitive Design
(2) Singular Points
(3) Zero Dynamics

Since the output function can be determined based on the physical interpretation, it is easy to design the controller. However, transformation of both input and coordination might produce singular points in the inverse of $\boldsymbol{\alpha}(\boldsymbol{q}_{eq})$. Plus, it is hard to figure out Zero Dynamics and obtain the ideal behavior, which is the significant key of this control.

4.2 Position Control along with Upward Posture

Output function vector is designed as

$$\boldsymbol{h} = \begin{bmatrix} x - (x_r + A(t)\cos(2\pi f t)) \\ y - (y_r - A(t)\sin(2\pi f t)) \\ z - z_r \end{bmatrix} \tag{43}$$

$$A(t) = -k\frac{2}{\pi}\tan^{-1}(\rho * (t-\delta)) + k, \;\; k = \frac{r_o}{2} \tag{44}$$

1$^{\text{st}}$ and 2$^{\text{nd}}$ output functions are meant for the spiral motion on $x-y$ plane and 3$^{\text{rd}}$ one is for the stabilization of z. Remark the linear subspace has 9 dimension and the nonlinear subspace has 6 dimension, because all of the relative degree for each of output function is 3.

4.3 Numerical Simulation

In this simulation, it is assumed that all of states can be measured by sensors. The important conditions of numerical simulation are shown in Table 3 and Table 4. The result of the output zeroing controller under these conditions is in Fig.3 and Fig.4. As is shown in Fig.4, the position tracking is obtained along with the stabilization of roll and pitch angle for upward attitude. The attitude, Zero Dynamics in this case, does not converge to upward posture, if the tracking path is designed to be straight. This means spiral motion has significantly positive effect to Zero Dynamics indirectly. However, stability analysis of

Table 4. Simulation Parameters

$\boldsymbol{\xi}_0$:	initial position $[x_0, y_0, z_0]^T$:	$[25.0, -25.0, 10.0]^T$ [cm]
$\boldsymbol{\eta}_0$:	initial attitude $[\phi_0, \theta_0, \psi_0]^T$:	$[50.0, -30.0, 0.0]^T$ [deg]
$\boldsymbol{\xi}_{r1}$:	landing position $[x_1, y_1, z_1]^T$:	$[0, 0, 0]^T$ [cm]
$\boldsymbol{\xi}_{r2}$:	landing position $[x_2, y_2, z_2]^T$:	$[20.0, 25.0, 30.0]^T$ [cm]
$\boldsymbol{\eta}_r$:	upward attitude $[\phi_r, \theta_r, \psi_r]^T$:	$[45, -35.3, *]^T$ [deg]

Zero Dynamics has not done due to the high dimensional dynamics unlike SISO case. Plus, it is also acknowledged that the convergence point depends on the initial condition. However, the position control is always guaranteed, which indicates the use for the failure case of other UAVs, since this enables collision avoidance, at least. Here, other remarks are mentioned. In addition to position control strategy, circle motion and safe landing are easy to be realized by some modification of the output function vector shown in (43).

5. CONCLUSION

In this paper, first, a new UAV model having three rotor inputs was suggested for the purpose of system analysis of three inputs system and design of the controller for the failure case. Next, the two types of nonlinear state equations were derived for model analysis and controller design. Then, the analysis of Euler angle model revealed that hovering control is mathematically impossible via three inputs. Reflecting the result of the limitation, output zeroing controller is designed along with upward posture, the desired Zero Dynamics. Finally, numerical simulation showed the performance. For the practical purpose, the contributions of this paper can be both the controller design for the failure case and the knowledge of the limitation of the mechanical design.

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